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TRANSPORT PROCESSES IN FRACTALS—I

CONDUCTIVITY AND PERMEABILITY OF A LEIBNIZ PACKING IN THE LUBRICATION LIMIT

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Abstract—The equivalent thermal resistance and permeability of a Leibniz packing with small gap between the disks are calculated in the lubrication approximation. Power laws are obtained for both processes. For conduction, the packing of the interstices yields an isotropic equivalent network, whatever the original configurations. This is related to the scale invariance of the resistances of the gaps, and is used to derive the exponent of the power law with an excellent precision. The equivalent permeability cannot be reduced to such a simple result, since the original configuration is always reminded; a good estimation of the exponent is derived by a simple argument.

1. INTRODUCTION

Recently most of our efforts have been devoted to spatially periodic media and have been condensed in a forthcoming book (Brenner & Adler 1985); such media can be viewed as adequate to represent some features of, in example, homogeneous porous materials. These media are supposed to possess a translational symmetry, i.e. they are globally invariant along sets of basic vectors. A cubic array of identical spheres furnishes the simplest example of such a medium.

Concomitantly, geometrical objects with a dilational invariance have attracted more and more attention. A loose definition of such an object is that it looks the same whatever the observation scale; the usual example is the classical Sierpinski gasket. Such objects which are called fractals were first introduced by Mandelbrot; note that the reader is referred only to his most recent book (Mandelbrot 1982). This subject has now grown up in a very active field, since it gives a basic framework for self-similar structures. These structures can be found in polymer coils, polymers adsorbed on a solid surface (De Gennes 1979), percolation network (Stauffer 1979); many suggestive examples are found in (Mandelbrot 1982).

The following recent contributions dealing with fractals and mostly with the Sierpinski gasket may be cited here. Critical phenomena, statistical mechanics and conductivity problems are presented in Gefen *et al.* (1980, 1981). The density of states and the spectrum of the Schrödinger equation have also attracted some attention (Alexander & Orbach 1982; Ramal & Toulouse, 1982).

However, to our knowledge, no attempt has been made yet to calculate the various properties of such a medium from a continuous point of view. It is the purpose of this series to fill in this gap. One of the many challenges could be the precise hydrodynamic calculation of a self-similar porous sphere as a model of a polymer coil.

With this total absence of previous contributions in mind, we start by the simplest fractal object which is a Leibniz packing, according to Mandelbrot's terminology. Such a packing is obtained in the following way. Let us draw three disks tangent one to the other one; in the interstices between them, a fourth disk tangent to the three former ones can be added. In the three new interstices which appear, three new disks may be added and so on... In the present case, the disks are assumed not to be exactly tangent. A gap is left between them; it is assumed to be small with respect to the radius of the disks between which it is located; it is a constant fraction of the radius of the smallest disk.

Besides its theoretical simplicity, such a packing can be considered as a basic model for

mixtures of particles of various dimensions. In this sense, it may be considered as a generalization of previous contributions related to homogeneous materials (Cone 1971; Batchelor & O'Brien 1977). Moreover it will serve as a useful limiting case in the rest of this series, and as a guide when basic assumptions on the behaviour of physical quantities are needed.

Two processes are analyzed in this fractal object. First, its equivalent resistance is calculated when the conductivity coefficient of the disks is assumed to be very large with respect to the conductivity of the continuous phase. Together with the hypothesis on the gap, this enables us to evaluate the transfer coefficient between two adjacent disks in the lubrication limit. Second, a Newtonian fluid is supposed to flow within the interstices and the equivalent permeability of the packing is calculated in the lubrication limit.

In section 2, the construction of the Leibniz packing is exposed and the various lubrication formulae of interest are recalled.

The third part is devoted to the description of the derivation of the numerical solution of the problem which involves two different steps: the construction of the packing with the insertion of disks and the calculation of the transport properties of the packing by "deleting" the disks.

Numerical results are presented and discussed in section 4. Some geometric features of Leibniz packings are first exposed, such as the evolution of the largest and smallest curvatures in each generation of disks. For both transport processes, power laws are obtained. However a major difference exists between them; the conduction process leads to the same equivalent resistance whatever the initial conditions are; in some sense, the packing becomes progressively isotropic as the number of inserted disks increases. This property enables us to calculate exactly the equivalent resistance of the packing. The convection process does not yield isotropy. For each set of initial conditions, particular coefficients for the power law are obtained; note that the exponent is a constant. As a check, isotropy can be derived under conditions reminiscent of conduction. Finally, the assumption that permeability is controlled by the smallest disks present in the packing leads to a fair estimation of the exponent of the power law.

Finally, the fractal and the spatially periodic characters are combined. This must be considered as an illustration of the fact that most porous media can be considered as homogeneous at a macroscopic length scale, but heterogeneous at a local scale. The global homogeneity is related to the global translational invariance at this scale, and is schematized by the spatially periodic character. The local heterogeneity is related to a local dilational invariance, which is in turn schematized by the fractal structure.

2. BASIC EQUATIONS

General

Consider three disks, almost tangent but otherwise arbitrary; the gap between any two of them is assumed to be small with respect to the radii of the two adjacent circles. They are numbered from 1 to 3.

In step $n = 1$, a new disk is inserted into the interstice between the three first disks. It is numbered as 4; the maximum value of its radius is $a_m(4)$. In order to leave a gap equal to $\epsilon a_m(4)$ where ϵ is very small with respect to 1, the actual value $a(4)$ of the radius is given by

$$a(4) = (1 - \epsilon)a_m(4) \qquad [1]$$

In step $n = 2$, three new disks are created in the three interstices (see figure 1a). This construction process can be continued in an obvious way. During the nth generation, 3^{n-1} new disks are introduced into the packing. Hence, the total number N_n of disks after n steps

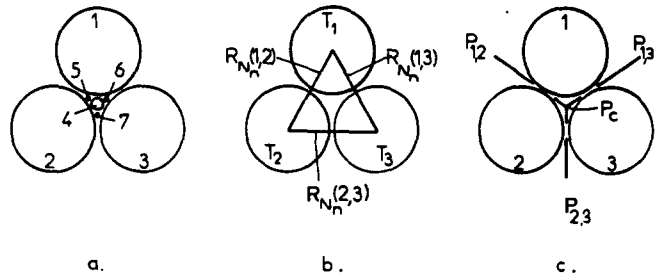


Figure 1. Leibniz packing. (a) The construction process is illustrated up to $n = 2$. The disk 4 is created during the step $n = 1$, while the disks $i = 5, 6, 7$ are created during the step $n = 2$. (b) For conduction, the packing may be viewed as an electrical network equivalent to a triangular system. (c) For convection, the packing is equivalent to a star electrical network.

is equal to

$$N_n = \frac{5 + 3^n}{2}. \qquad [2]$$

Of course, relation [1] is assumed to hold for any disk i.

In the conduction process, the conductivity k_p of the disks is assumed to be much larger than the conductivity k_f of the continuous phase; as an immediate consequence each disk i' has a constant temperature $T(i)$. The heat flux $q(i,i')$ between two adjacent disks i and i' is proportional to the temperature difference $T(i) - T(i')$. It will be shown in section 3 that we can reduce the corresponding electrical network to an equivalent triangular system as shown in figure 1(b), with resistances $R_N(i,i')$. Hence, the rates of heat flow $Q_N(i,i')$ per unit depth between the three initial disks are related to the temperature differences by

$$T(i) - T(i') = R_N(i,i')Q_N(i,i')/\pi k_f \quad (i,i' = 1, 2, 3). \qquad [3]$$

We wish to calculate the three resistances $R_N(1,2)$, $R_N(1,3)$, $R_N(2,3)$ as a function of the number of filling disks N_n inside the initial interstice and of the radii of the initial disks.

In the convection process, three different pressures are applied outside the three disks (see figure 1c). In general, the packing may be viewed as a star electrical network (note that this is obvious at step $n = 0$, when only three disks are present; the network is also equivalent to a triangle, but we prefer to view it as a star since it is reminiscent of the original configuration). The pressure at the center of the equivalent star is denoted by p_c. The flow rates per unit depth $\bar{q}(i,i')$ of fluid flowing inside the gap (i,i') are proportional to the pressure difference

$$\bar{q}(i,i') = a^2(1)\, P_N(i,i')\,(p(i,i') - p_c) \Big/ \left(\frac{9\pi}{\sqrt{2}}\mu\right) \quad (i,i' = 1, 2, 3) \qquad [4]$$

where μ is the viscosity of the fluid.

Again, we wish to calculate the three coefficients $P_N(1,2)$, $P_N(1,3)$ and $P_N(2,3)$ as a function of the number of filling disks and of the radii of the initial disks.

In the rest of this section, we shall first calculate the characteristics of a newly introduced disk as a function of the surrounding ones and then recall the lubrication formulae.

Geometry

Let us consider three disks centered at the points A, B, C and of radii a_A, a_B and a_C (see figure 2). The circles are external one to the other one and are separated by three gaps

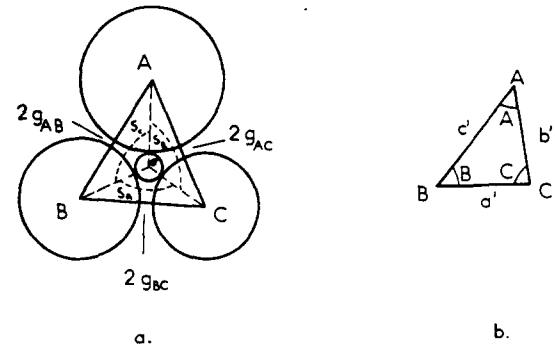


Figure 2. Geometry of the packing (a). Some notations are given in (b).

respectively equal to $2g_{AB}$, $2g_{BC}$ and $2g_{AC}$. We first want to know the radius a_s of the inner circle tangent to the three circles A, B and C and centered at the unknown points.

When the three initial circles are tangent, the problem was algebrically solved by Soddy (1936) (see also Coxeter 1969). The generalization to the present situation is not difficult and is briefly given in the appendix for sake of completeness. It is preferable to use curvatures instead of radii; they are defined by:

$$\alpha = 1/a_A, \quad \beta = 1/a_B, \quad \gamma = 1/a_C, \quad \sigma = 1/a_S, \tag{5}$$

σ is the larger root of the second degree equation

$$\begin{aligned}&\sigma^2\,(l^2+m^2+n^2-2lm-2mn-2ln+4lmn)+2\sigma[\alpha l^2+\beta m^2+\gamma n^2-\ldots.\\&-(\alpha+\beta)lm-(\alpha+\gamma)ln-(\beta+\gamma)mn]+\alpha^2l^2\\&+\beta^2m^2+\gamma^2n^2-2\alpha\beta lm-2\alpha\gamma ln-2\beta\gamma mn=0\end{aligned} \tag{6}$$

Where l, m, n are given by

$$l = (1+g_{BC}\beta)(1+g_{BC}\gamma) \tag{7a}$$

$$m = (1+g_{CA}\gamma)(1+g_{CA}\alpha) \tag{7b}$$

$$n = (1+g_{AB}\alpha)(1+g_{AB}\beta) \tag{7c}$$

It may be verified that (6) correctly reduces to Soddy's formula (1936), when the three initial circles are tangent, i.e. when l, m, n are equal to 1.

Finally, it should be recalled from (1) that the disk which is actually inserted is only a fraction $1-\epsilon$ of its maximum size a_s; hence, assuming that the disk number is i

$$a(i) = (1-\epsilon)a_S \tag{8}$$

The gap is thus equal to ϵa_s.

Lubrication formulae

When the conductivity k_p of the disks is supposed to be very large with respect to the conductivity k_f of the continuous medium and when the gap $2g(i,i')$ between two disks of radii $a(i)$ and $a(i')$ is very small with respect to these radii, the lubrication approximation

can be used. The temperature difference is then related to the heat flux per unit depth by

$$T(i) - T(i') = r(i, i') \cdot q(i, i')/\pi k_f \qquad [9]$$

where

$$r(i, i') = [g(i, i') \cdot (1/a(i) + 1/a(i'))]^{1/2}, \qquad [10]$$

$r(i, i')$ denote the thermal resistances.

Second, consider a Newtonian fluid forced through the gap between two parallel cylinders. The flow rate per unit length may be expressed as a function of the pressure difference by

$$\bar{q}(i, i') = P'(i, i') \cdot \frac{\Delta p}{\frac{9\pi}{\sqrt{2}} \cdot \mu} \qquad [11]$$

where the equivalent permeability $P'(i, i')$ is given by

$$P'(i, i') = 4\sqrt{2} \cdot [g^5(i, i') \cdot (1/a(i) + 1/a(i'))]^{1/2}. \qquad [12a]$$

A dimensionless permeability $P(i, i')$ may be defined as

$$P(i, i') = P'(i, i')/a^2(1) \qquad [12b]$$

It is thus equivalent to consider that the disk 1 has a radius equal to 1, which will be implicity assumed in the following.

3. METHOD OF SOLUTION

In view of the apparent complexity of the basic equations [6], [10] or [12], the solution was numerically calculated.

First, the geometry of the system was determined. Starting from three arbitrary disks, the packing was constructed in a few steps. Typically, eight steps were generated with a total number of disks equal to 3283 (see [2]). An example after 2 steps is given in figure 3(a).

Simultaneously, for a given transport process, the resistance and the permeability of each gap are calculated by the relations [10] and [12] respectively. The equivalent networks are shown in figures 3(b) and (c), for $n = 2$.

It is well known from the theory of electrical networks (Bollobàs 1979) that a star is equivalent to a triangle. The star with resistances A, B, C may be replaced by the triangle with resistances A', B', C' (see figure 3d) if

$$A' = S/A, \quad B' = S/B, \quad C' = S/C \qquad [13]$$

where

$$S = AB + BC + CA.$$

This transformation may be inverted. The triangle with resistances A', B', C' may be replaced by a star with resistances A, B, C if

$$A = B'C'/T, \quad B = C'A'/T, \quad C = A'B'/T \qquad [14]$$

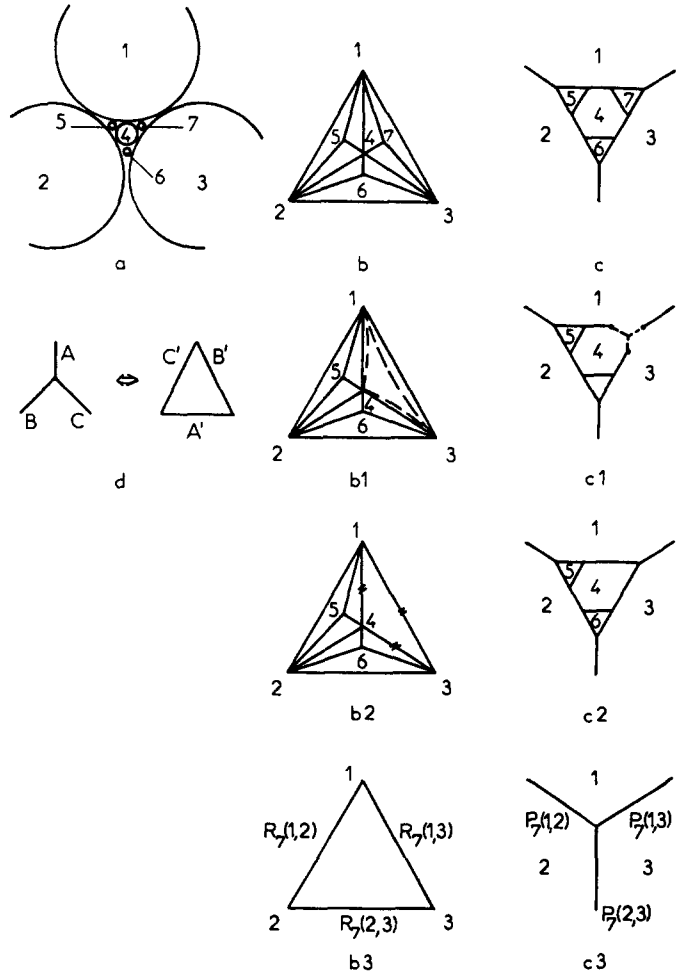


Figure 3. Equivalent networks for the Leibniz packing. The basic geometry is recalled in (a). The equivalent network for conduction (respectively convection) are shown in (b) (respectively c). The simplification of these graphs is explained in the text. The star-triangle transformation is illustrated in (d): the letters A, B, C, A', B', C' denote the resistances.

where

$$T = A' + B' + C'.$$

The equivalent network in both cases can be obtained by a repeated application of formulae [13] or [14]. For instance, in the conduction case, the star relating the disk 7 with its neighbours 1, 3 and 4 (see figure 3b) is replaced by an equivalent triangle (see the broken lines in figure 3b1). The resulting graph can be further simplified by application of the classical formula about resistances disposed in parallel. Hence, we obtain the equivalent network (see figure 3b2) where the disk 7 has been "deleted" and the new values $r'(4, 1)$, $r'(4, 3)$ and $r'(1, 3)$ of the resistances (4,1), (4,3) and (1,3) modified as

$$r'(4, 1) = (r(4, 1)^{-1} + r(7, 3)/S_1)^{-1}$$

$$r'(4, 3) = (r(4, 3)^{-1} + r(7, 1)/S_1)^{-1}$$

$$r'(1, 3) = (r(1, 3)^{-1} + r(4, 7)/S_1)^{-1}$$

with

$$S_1 = r(7,3)\cdot r(7,1) + r(7,1)r(7,4) + r(7,4)r(7,3)$$

by application of [13].

This "deletion" is systematically performed until only the three first disks remain; the resistances of the various branches are $R_7(1,2)$, $R_7(2,3)$ and $R_7(3,1)$ according to [3] (see figure 3b3).

The same process may be applied to the convection network. Here, triangles are transformed into stars and the resulting graph is simplified by adding resistances in series. Again, this is systematically performed until only the first three disks remain. According to [4], the permeabilities of the various branches are $P_7(1,2)$, $P_7(2,3)$ and $P_7(3,1)$ (see figure 3.c.3).

These calculations were done by computer. Each step of the calculations was carefully checked on several examples.

4. RESULTS AND DISCUSSION

Geometry

We shall start this section by a brief discussion of the geometrical characteristics of the packing.

Three basic geometries are investigated and are described in Table 1. Four values of the parameter ϵ are systematically studied

$$\epsilon = 10^{-2},\quad 10^{-3},\quad 10^{-4},\quad 10^{-5}. \qquad [15]$$

$n = 8$ successive generations of disks were found to be sufficient to display the desired behaviour.

Let us first consider a close packing of disks obtained for $\epsilon = 0$ and let us study the evolution of the largest and smallest curvatures $S_{max}(n)$ and $S_{min}(n)$ of the disks generated during the step n.

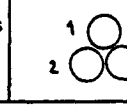
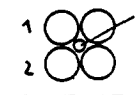
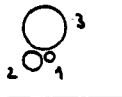
As already stated, this corresponds to the case where l, m and n are equal to 1. Hence, the second degree equation [6] may be somewhat simplified; the curvature of the circle internally tangent to the three initial circles may be expressed as (Soddy 1936)

$$\sigma = \alpha + \beta + \gamma + 2d \qquad [16a]$$

where d is given by the relation

$$d = (\alpha\beta + \beta\gamma + \gamma\alpha)^{1/2}. \qquad [16b]$$

Table 1. Geometry. Names for the various cases are derived from the fact that three equal disks can form the basic pattern of an hexagonal array and so on... The values of the gap ϵ are given by [15]. The radius of disk 1 is assumed to be equal to $1-\epsilon/2$ in all cases (see [12])

Name	Hexagonal	Square	Triangular
Initial radii	$1-\frac{\epsilon}{2}$, $1-\frac{\epsilon}{2}$, $1-\frac{\epsilon}{2}$	$1-\frac{\epsilon}{2}$, $1-\frac{\epsilon}{2}$, $\sqrt{2}-1-\frac{\epsilon}{2}$	$1-\frac{\epsilon}{2}$, $2-\frac{\epsilon}{2}$, $3-\frac{\epsilon}{2}$
Patterns			

Let us apply these formulae (16) to the construction of a close packing of disks for $\epsilon = 0$. Since σ is a monotonically increasing function of α, β and γ, the smallest disks (i.e. the largest curvatures) are found in the interstices between the smallest disks of the last three steps (see figure 4). Hence, as an application of [16], $S_{max}(n)$ may be expressed as

$$S_{max}(n) = S_{max}(n-3) + S_{max}(n-2) + S_{max}(n-1) + 2d(n) \qquad [17a]$$

with

$$d(n) = [S_{max}(n-3)S_{max}(n-2) + S_{max}(n-2)S_{max}(n-1) + S_{max}(n-1)S_{max}(n-3)]^{1/2} \qquad [17b]$$

This may be equivalently represented by Boyd's matrix notation

$$(S_{max}(n-3), S_{max}(n-2), S_{max}(n-1), d(n)) = (\alpha, \beta, \gamma, d) \cdot \mathbf{S}^{n-1}, \quad \alpha \leqslant \beta \leqslant \gamma \qquad [18a]$$

with

$$\mathbf{S} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{pmatrix} \qquad [18b]$$

Observe how the two first columns of the matrix S make the curvatures shift. The matrix S may be diagonalized and written as

$$\mathbf{S} = \mathbf{P} \cdot \begin{pmatrix} \lambda_0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix} \cdot \mathbf{P}^{-1} \qquad [19a]$$

where the columns of the matrix P are the components of the eigenvectors of S. Hence, (18) may be evaluated as

$$\mathbf{S}^{n-1} = \mathbf{P} \cdot \begin{pmatrix} \lambda_0^{n-1} & 0 & 0 & 0 \\ 0 & \lambda_1^{n-1} & 0 & 0 \\ 0 & 0 & \lambda_2^{n-1} & 0 \\ 0 & 0 & 0 & \lambda_3^{n-1} \end{pmatrix} \cdot \mathbf{P}^{-1} \qquad [19b]$$

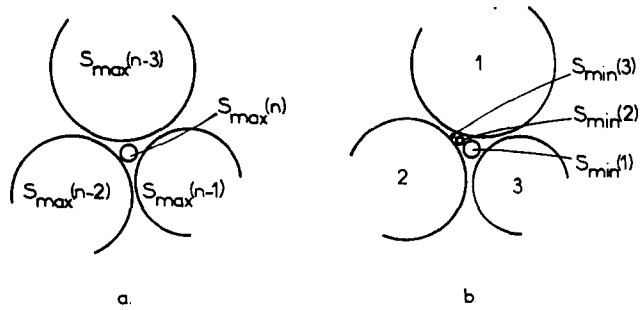


Figure 4. Construction of the largest curvature (a); the disk(s) of maximum curvature $S_{max}(n)$ is found inside the interstice located between the disks with maximum curvature at the three previous steps $S_{max}(n-3)$, $S_{max}(n-2)$, $S_{max}(n-1)$. Construction of the smallest curvature (b); the disk(s) is obtained by progressive insertion of disks inside the gap (1,2).

For large values of n, the largest eigenvalue λ_0 of S becomes predominant and we can write

$$S_{max}(n) \alpha \lambda_0^n \alpha N_n^{\log \lambda_0/\log 3} \qquad [20a]$$

with the understanding that λ_0^{-1} is included in the proportionality coefficient.

The value of the last exponent is equal to 0.96600, since

$$\lambda_0 = 2.8900 \qquad [20b]$$

On the other hand, the largest disks at a given step are obtained by the progressive insertion of disks inside the gap between two of the initial disks, as it is illustrated in figure 4(b). This is again due to the fact that σ is a monotically increasing function of α, β and γ (see [16]). A matrix formulation analogous to [18] can be given; however, all the eigenvalues of the matrix are equal to 1. Thus, the minimal curvature $S_{min}(n)$ is not given by a power law anymore, but by a polynomial (see Boyd 1973)

$$S_{min}(n) = (\alpha + \beta)n^2 + 2dn + \gamma. \qquad [21]$$

When ϵ is not equal to zero, the curvatures of all the disks are numerically obtained by a repeated application of the formulae [6] and [7]. The influence of the parameter ϵ on $S_{min}(n)$ and $S_{max}(n)$ is illustrated in figure 5 for the hexagonal array. For $\epsilon = 0$, the analytical results [18] and [21] are reobtained; the exponent of $S_{max}(n)$ in [20] equal to 0.96600 is also well verified. It was confirmed that, for the range of values of ϵ, the smallest and largest disks for $\epsilon \neq 0$ were found at the same place in the packing as for $\epsilon = 0$.

Finally, note that the log–log plot in figure 5 makes the difference between $\epsilon = 0$ and 10^{-2} appear very small, which is quite not the case, especially for $S_{min}(n)$. Note also the difference in behaviour; the largest curvatures differ only by a constant in a first approximation, while the smallest curvatures separate progressively.

The insertion of smaller and smaller disks may create some problem in view of the lubrication approximation. If a disk is inserted inside the lubrication zone of two disks generated at a previous step, the approximation does not hold anymore.

At first sight, it may be thought that the critical step is related to the smallest disks. This is not the case since in view of [20], the radius ratio between the inner disk and the external ones ranges between 2.89 and 24.1; thus the lubrication zones cannot be disturbed.

Actually, the critical point is connected with the largest disks as shown in figure 4(b); these disks penetrate progressively inside the lubrication zone of the initial disks 1 and 2. For

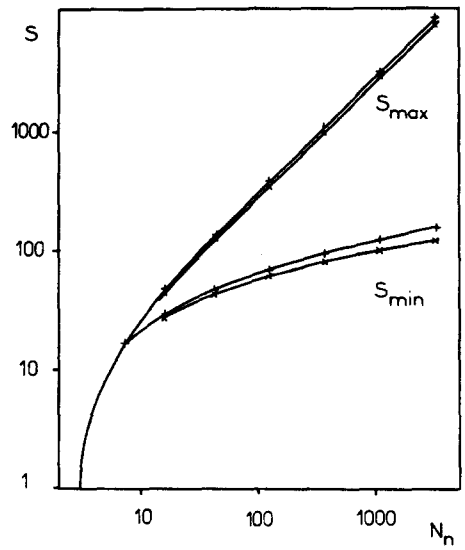


Figure 5. Largest and smallest curvatures for an hexagonal array as a function of the number of disks N_n. Values of ϵ are: 0(+), 10^{-2}(×).

the hexagonal array, when ϵ is varying from 0 to 10^{-2}, it is easy to show that the distance between the line of centers of the initial disks 1 and 2 and the center of the disk $S_{min}(8)$ varies from 0.11 to 0.08. This is acceptably outside the lubrication zone of disks 1 and 2.

Moreover, it will be demonstrated in the following that the transfer controlled by the initial gaps is largely negligible with respect to the transfers controlled by the other gaps. However, for $\epsilon \neq 0$, the insertion process cannot be indefinitely continued since, for n large enough, a disk is created which is located on the line of centers of the original disks. This limitation is not important since the power law behaviour is obtained before this phenomenon occurs.

Conductivity

Calculations were systematically performed for the configurations listed in table 1.

The presentation of the results may be started by the hexagonal configuration, which is given in figure 6. The equivalent resistances of the branches are all equal to $R_N(1,2)$ (see [3]) since the initial disks are identical. The first striking feature is that a power law is very quickly reached; for $N_n = 16$ disks, i.e. after 3 steps, $R_N(1,2)$ is very close to the final power law, which may be expressed as

$$R_N(1,2) = KN^{\alpha} \qquad [22]$$

where K is a constant which depends upon ϵ, and α is the exponent which can be deduced from the numerical results as equal to

$$\alpha \simeq -0.464. \qquad [23]$$

This result is of course consistent with our intuition, since such a fractal structure was supposed to lead necessarily to a power law. The second important feature is that we obtain within a very good approximation the following relation for various values of ϵ

$$R_N(1,2)/\sqrt{\epsilon} = \text{cst, for a fixed value of } N. \qquad [24]$$

The power 1/2 in this relation is obviously a direct consequence of the fact that the resistance of the elementary gap varies as a square root of the gap (see [10]).

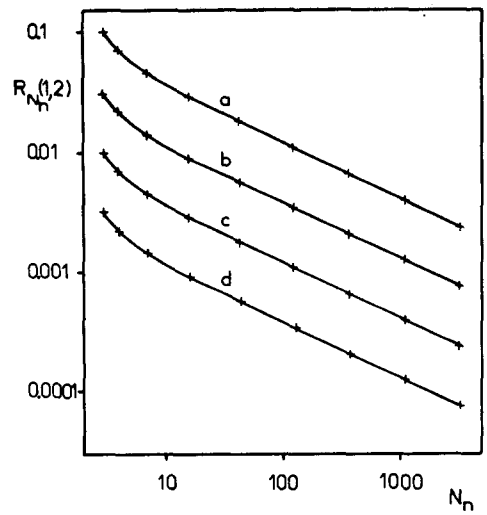


Figure 6. Equivalent thermal resistances $R_N(1,2)$ for an hexagonal array as a function of the number of disks N_n, values of ϵ are: 10^{-2}(a), 10^{-3}(b), 10^{-4}(c), 10^{-5}(d).

It is surprising that the relation [24] holds for all the values of N_n. In view of figure 5, one would actually expect some differences for large N_n, when the geometrical structure of the packing is a sensitive function of ϵ.

A pleasant consequence of this property is that all the curves for the various values of ϵ can be deduced from a master curve with a very good approximation:

$$R_N(1,2) = \sqrt{\epsilon} R_N(1,2)^\circ \qquad [25]$$

where $R_N(1,2)^\circ$ is tabulated in table 2.

The same may be shown to be true for a square and a triangular packing. The corresponding quantities $R_N(1,2)^\circ$, $R_N(2,3)^\circ$ and $R_N(1,3)^\circ$ are also given in table 2.

Let us now compare in table 2 the three equivalent resistances of the triangular network. It is remarkable to notice that the values tend towards a common limit, though the initial values are largely different due to the variations in the original configurations (illustrated in table 1). Moreover, the resistances of the two other packings (hexagonal and square) also tend towards the same limit, as it can be observed from table 2.

Hence, as a consequence of the numerical results, the following empirical law may be stated. In the limit of large values of N_n, the resistances associated to the various branches

Table 2. Reduced thermal resistances $R_N(i,i')^\circ$ for the various arrays. $R_N(i,i')^\circ$ is defined by [25]. The isotropic case corresponds to [31]

	Hexagonal	Square		Triangular			Isotropic
N_n	$R_N(1,2)^\circ$	$R_N(1,2)^\circ$	$R_N(1,3)^\circ = R_N(2,3)^\circ$	$R_N(1,2)^\circ$	$R_N(2,3)^\circ$	$R_N(1,3)^\circ$	
3	1	1	0.131	0.866	0.645	0.817	1.414
4	0.695	0.686	0.837	0.633	0.498	0.603	0.849
7	0.456	0.450	0.514	0.430	0.361	0.415	0.509
16	0.289	0.286	0.311	0.278	0.247	0.272	0.305
43	0.179	0.178	0.187	0.175	0.162	0.172	0.183
124	0.109	0.109	0.112	0.108	0.103	0.107	0.110
367	0.0665	0.0663	0.0676	0.0659	0.0640	0.0655	0.0660
1096	0.0402	0.0401	0.0406	0.0400	0.0392	0.0398	0.0396
3283	0.0242	0.0242	0.0244	0.0241	0.0239	0.0241	0.0238

of a Leibniz packing are equal and behave like a unique power law irrespective of the initial conditions.

This seems to be the salient feature of the process since it will now be shown that the exponent α of the law may be derived from this hypothesis of isotropy.

This derivation may be made as follows. Let us consider a Leibniz packing with $n+1$ steps, where n is sufficiently large. The deletion of the generations $n+1, n, \ldots, 2$, by the process previously described yield the situation depicted in figure 7. According to the recurrence hypothesis that we have just made, the resistances $R_N(i, i')$ of the solid lines are equal one to the other one and given by [22]

$$R_{N_n}(i, i') = KN_n^{\alpha} \qquad [26]$$

since it corresponds to a Leibniz packing with n generations. The resistances of the initial gaps (the broken lines in figure 7) can be neglected since they are very large when compared to the resistances R_{N_n} when n is large. It is then a simple matter to show that the network is equivalent to a triangular network whose links have a resistance equal to $3/5\ R_{N_n}$. Hence,

$$R_{N_{n+1}} = \left(\frac{3}{5}\right) R_{N_n} \qquad [27a]$$

or equivalently

$$R_{N_n} \alpha \left(\frac{3}{5}\right)^n \qquad [27b]$$

But n may be expressed as a function of N_n as a direct application of [2]. Thus, [27b] may be modified as

$$R_{N_n} \alpha N_n^{\alpha'} \qquad [28]$$

where the exponent α' is given by

$$\alpha' = \frac{\log 3/5}{\log 3} = -0.46497 \qquad [29]$$

in excellent agreement with the empirical value [23].

Note that in triangular systems with dilational invariance (Gefen *et al.* 1980), $\log(d+1)$ appears in most of the results; d is here the dimension of space. In the present configuration which is planar, d is equal to 2 and log 3 appears in [29].

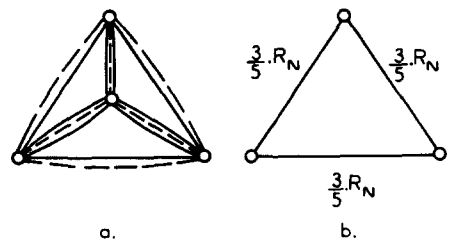


Figure 7. Derivation of the thermal power law [22]. (a) Packing with $n+1$ steps; the solid lines correspond to the equivalent resistances of the last steps; the broken lines are the initial resistances. (b): resistances equivalent to the initial packing of $n+1$ steps.

The success of this semi-empirical result suggests that we should obtain a good model of the packing by assuming that the resistances of all the branches of the network are equal. Since the gap between two disks is a constant fraction ϵ (see [8]) of the radius of the smaller disk i, the resistance may be expressed as

$$\sqrt{\frac{\epsilon}{2}\left(1+\frac{a(i)}{a(i')}\right)} \qquad [30]$$

with $a(i) < a(i')$.

Let us assume that the ratio $a(i)/a(i')$ is negligible; the network is thus only composed of resistances equal to $\sqrt{\epsilon/2}$.

Some easy algebra leads to the equivalent resistance as a function of n. First, a recurrence formula may be exactly calculated as

$$R_{N_{n+1}} = \frac{1}{\dfrac{1}{R_{N_n}} + \dfrac{1}{3t_n}} \qquad [31]$$

where t_n is given by

$$t_n = \sqrt{\frac{\epsilon}{2}} \cdot \left(\frac{3}{5}\right)^n. \qquad [32]$$

The equivalent resistance R_{N_n} is then deduced as

$$R_{N_n} = \sqrt{2\epsilon} \cdot \left[1+\left(\frac{5}{3}\right)^n\right]^{-1}. \qquad [33]$$

Hence, when n is sufficiently large, R_{N_n} is equivalent to

$$R_{N_n} = \sqrt{2\epsilon} \cdot \left(\frac{3}{5}\right)^n. \qquad [34]$$

This is the so-called isotropic case, since all the elementary resistances are assumed to be equal. Comparison is made in table 2 and agreement is seen to be excellent. Hence, the numerical results are well explained by this semi-empirical way of reasoning.

In the following (see [40] and sq.), it will be suggested that the fundamental reason for the success of this simple model is related to the scale invariance of the resistance of the gaps. That is to say, the gap resistance is unchanged when all dimensions are multiplied by a unique factor α. This invariance leads to the observed isotropy and thus to the particular form of the law.

Finally, the fractal and the spatially periodic characters may be combined. Consider an hexagonal spatially periodic pattern of disks of radius $1-\epsilon/2$, or a square one. The interstices between them are then filled up in the previously described manner. The result may be regarded as a basic model of an heterogeneous material which is translationally invariant.

The resulting arrays are isotropic; hence the macroscopic conductivity tensor or its inverse the macroscopic resistance tensor is spherical; note that the general properties of these tensors are given in Brenner & Adler (1985). The value of the macroscopic resistance

coefficient may be expressed as

$$\text{hexagonal} \quad R_{N_n}(1,2)/2\sqrt{3} \tag{35a}$$

$$\text{square} \quad 1/2[1/R_{N_n}(1,2) + 1/R_{N_n}(1,3)] \tag{35b}$$

Introduction of [34] into [35] yields

$$\text{hexagonal} \quad \sqrt{\frac{\epsilon}{6}} \cdot \left(\frac{3}{5}\right)^n \tag{36a}$$

$$\text{square} \quad \frac{1}{2}\sqrt{\frac{\epsilon}{2}} \cdot \left(\frac{3}{5}\right)^n \tag{36b}$$

which may be interesting to check experimentally.

Permeability

Numerical calculations were performed for the geometries given in table 1.

Results for the hexagonal array are represented in figure 8. As for conduction, the equivalent permeability $P_N(1,2)$ may be expressed by a power law

$$P_N(1,2) = KN_n^{\beta} \tag{37}$$

where K depends upon ϵ, while β is deduced to be close to -2

$$\beta \simeq -2.04 \tag{38}$$

In view of [24], it was *a priori* expected that, for a given value of N,

$$P_N(1,2) \cdot \epsilon^{5/2} = \text{cst} \tag{39}$$

However, the complete numerical results show that the relation [39] is not well verified

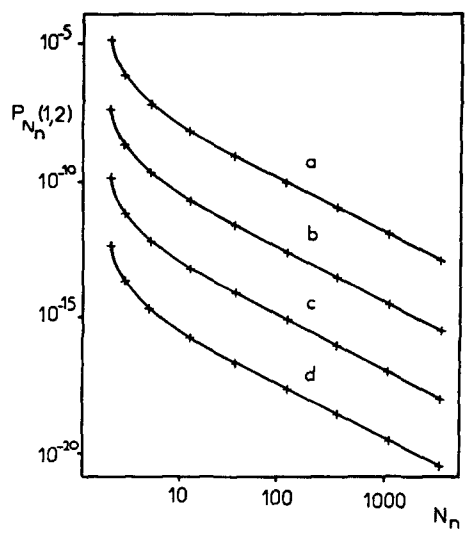


Figure 8. Equivalent permeabilities for an hexagonal array as a function of the number of disks N_n. Values of ϵ are: 10^{-2}(a), 10^{-3}(b), 10^{-4}(c), 10^{-5}(d).

except when ϵ is very small. This is of course due to the stiff dependance of the permeability of the elementary gap as a function of the width of the gap; according to [12], it varies as $\epsilon^{5/2}$. Hence, the constant K is a sensitive function of the geometry of the packing.

The second disturbing feature is that for various original configurations we do not obtain the same final results as for conductivity. This point is illustrated in table 3. Hence, the isotropy hypothesis is not valid here, and we cannot deduce the value of the exponent β as before.

This behaviour was thought to be caused by the non scaling invariance of the permeability $P(i,i')$ of the gap (i,i') as it is expressed by [12]. When all the dimensions are multiplied by a factor α, the permeability is multiplied by α^2, as it is easily derived from [12]. In order to check the fact that isotropy is not obtained except for $\nu = 1$, the following fictitious law of permeability was introduced

$$P(i,i') = 2^{\nu/2} \cdot \left[g^{\nu}(i,i') \cdot \left(\frac{1}{a(i)} + \frac{1}{a(i')} \right) \right]^{1/2} \qquad [40]$$

where all quantities are dimensionless.

Again a power law was obtained for the equivalent resulting permeability $P_N(i,2)$. Let n denote the exponent of this power law; n is represented as a function of ν in figure 9.

A useful check of the numerical calculations is provided by the case $\nu = 1$. For this value, the isotropic character is again obtained as it is expected on intuitive grounds. Moreover, following the same line as previously, the exponent ν is found to be equal to $-\log 3/5/\log 3$, and this is well verified numerically.

We are now left with the question of evaluating the exponent. A very simple argument may be derived as follows. When ν is large enough, it may be assumed that the equivalent permeability is controlled by the smallest gaps. Hence, the introduction of [8] into [40] approximately yields

$$P_N \sim \epsilon^{\nu/2} \cdot [S_{\max}(N)]^{-(\nu-1)/2} \qquad [41]$$

As a direct consequence of [20], P_N is obtained as

$$P_N \sim \epsilon^{\nu/2} \cdot N_n^{-0.483(\nu-1)}. \qquad [42]$$

Table 3. Equivalent permeabilities for the triangular array. $\epsilon = 10^{-2}$

N_n	$P_{N_n}(3,2)$	$P_{N_n}(3,1)$	$P_{N_n}(2,1)$
3	$0.91386.10^{-5}$	$0.11571.10^{-4}$	$0.12273.10^{-4}$
4	$0.17718.10^{-5}$	$0.19347.10^{-5}$	$0.19862.10^{-5}$
7	$0.22732.10^{-6}$	$0.18541.10^{-6}$	$0.17320.10^{-6}$
16	$0.25912.10^{-7}$	$0.19844.10^{-7}$	$0.18170.10^{-7}$
43	$0.29024.10^{-8}$	$0.21900.10^{-8}$	$0.19946.10^{-8}$
124	$0.32470.10^{-9}$	$0.24393.10^{-9}$	$0.22179.10^{-9}$
367	$0.36327.10^{-10}$	$0.27252.10^{-10}$	$0.24765.10^{-10}$
1096	$0.40645.10^{-11}$	$0.30477.10^{-11}$	$0.27690.10^{-11}$
3283	$0.45477.10^{-12}$	$0.34094.10^{-12}$	$0.30974.10^{-12}$

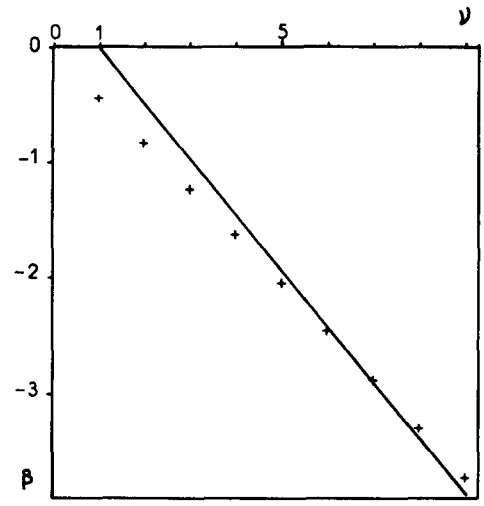


Figure 9. Variations of the permeability exponent β as a function of ν (see [40]). Points are obtained from the numerical results; the straight line is deduced from [43].

Hence β is given by

$$\beta = -0.483 \cdot (\nu - 1). \qquad [43]$$

The agreement when $\nu = 5$ is excellent, since we would obtain

$$\beta = -1.932 \qquad [44]$$

to be compared to [38].

But this excellent agreement seems fortuitous in view of the comparison made in figure 9.

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APPENDIX

The famous Greek mathematician Appolonius of Perga devised an algorithm to draw the five circles tangent (externally or internally) to three given circles. We are only interested by the inner circle (see figure 2).

Consider a triangle ABC detailed in figure 2(b). The half perimeter S of this triangle is given by:

$$S = \frac{1}{2}(a' + b' + c') \qquad [A1]$$

where a', b' and c' are the lengths of the three sides. From elementary geometry, it can be shown that

$$\cos^2 \frac{A}{2} = \frac{s(s-a')}{b'c'} \qquad [A2a]$$

$$\sin^2 \frac{A}{2} = \frac{(s-b')(s-c')}{b'c'} \qquad [A2b]$$

$$\cos A = \frac{b'^2 + c'^2 - a'^2}{2b'c'} \qquad [A2c]$$

$$\frac{a'}{\sin A} = \frac{b'}{\sin B} = \frac{c'}{\sin C} \qquad [A2d]$$

In view of [A2d], a', b' and c' may be replaced by $\sin A$, $\sin B$ and $\sin C$ in (A2c). It yields

$$\sin^2 A - \sin^2 B - \sin^2 C + 2\cos A \cdot \sin B \cdot \sin C = 0 \qquad [A3]$$

An identity which is valid for any triangle, or for any triplet of angles whose sum is equal to 2π. Going back to the general problem depicted in figure 2(a), the relations [A2a] and [A2b] can be written for the angle S_A in the triangle SBC.

$$\sin^2 \frac{S_A}{2} = \frac{(a_C + g_{BC})(a_B + g_{BC})}{(a_S + a_B)(a_S + a_C)},$$

$$\cos^2 \frac{S_A}{2} = \frac{(a_S + a_B + a_C + g_{BC})(a_S - g_{BC})}{(a_S + a_B)(a_S + a_C)}. \qquad [A4]$$

Equivalent relations may be written down for the angles S_B and S_c in the triangles SAC and SAB. Since the sum of the three angles $S_A/2$, $S_B/2$ and $S_c/2$ is equal to π, they may be inserted in the identity [A3], with $A = S_A/2$, $B = S_B/2$, $C = S_c/2$.

Introduction of the definitions [5] and [7], and elementary manipulations of [A3]

expressed for $S_A/2$, $S_B/2$ and $S_C/2$ yield

$$(\alpha + \sigma)\, l - (\beta + \sigma)m - (\gamma + \sigma)n + 2[(\sigma\gamma + \beta\gamma + \beta\sigma + g_{BC}\beta\gamma\sigma) \cdot (1 - g_{BC}\sigma) \cdot mn]^{1/2} = 0 \qquad [A5]$$

This equation is first squared and subsequently rendered symmetric by the use of [7a]; [6] easily follows.